On the Evolutionary of Bloom Filter False Positives - An Information Theoretical Approach to Optimizing Bloom Filter Parameters

Zhuochen Fan, Gang Wen, Zhipeng Huang, Yang Zhou, Qiaobin Fu, Tong Yang, Member, IEEE
Alex X. Liu, Fellow, IEEE and Bin Cui, Senior Member, IEEE

Abstract—The fundamental issue of how to calculate the false positive probability of widely used Bloom Filters (BF), from which the conventional wisdom is to derive the optimal value of \( k \), remains elusive. Since Bloom gave the false positive formula in 1970, in 2008, Bose et al. pointed out that Bloom’s formula is flawed; and in 2010, Christensen et al. pointed out that Bose’s formula is also flawed and gave another formula. Although Christensen’s formula is perfectly accurate, it is time-consuming and impossible to calculate the optimal value of \( k \). Based on the following observation: for a BF with \( m \) bits and \( n \) elements, if and only if its entropy is the largest, its false positive probability is the smallest, we propose the first approach to calculating the optimal \( k \) without any false positive formula. Furthermore, we propose a new and more accurate upper bound for the false positive probability. When the size of a Bloom Filter becomes infinitely large, our upper bound turns equal to the lower bound, which becomes Bloom’s formula and deepens our understanding towards it. Besides, we derive the bounds of correct rate of Counting Bloom Filters (CBFs) by applying our proposed formulas about BFs to them.

Index Terms—Bloom Filter, false positive, information entropy, compression, upper bound, lower bound, Counting Bloom Filter

1 INTRODUCTION

1.1 Motivation

A Bloom Filter (BF) is a compact data structure used for quickly checking whether an element belongs to a set or not [1]. Given a set \( S \) of \( n \) elements, we create a bit array \( A \) of length \( m \) as follows. First, we initialize each bit of \( A \) to 0, then for each element \( x \in S \), we use \( k \) hash functions to compute \( k \) hash values: \( h_1(x), h_2(x), \ldots, h_k(x) \) where each hash value is in the range \([1, m]\). Second, for each \( 1 \leq i \leq k \), we let \( A[h_i(x)] = 1 \). The resulting bit array \( A \) is called the BF for set \( S \). To query whether \( y \in S \), we first use the same \( k \) hash functions to compute \( k \) hash values: \( h_1(y), h_2(y), \ldots, h_k(y) \). Second, we check whether the corresponding \( k \) bits in \( A \) are all 1s (i.e., whether \( A[h_1(y)] \land A[h_2(y)] \land \cdots \land A[h_k(y)] = 1 \) holds); if yes, then \( y \in S \) may probably hold and we can further check whether \( y \in S \); if no, then \( y \in S \) definitely does not hold. The cases that the BF shows that \( y \in S \) may hold but actually \( y \notin S \) are called false positives (FP). The FP probability \( f \) can be calculated from \( n, k, \) and \( m \).

As set membership query is a fundamental operation in many applications [2], and BFs have the advantages of small memory consumption, fast query speed, and no false negatives, BFs have been widely used in networks, databases, data mining and analysis, and machine learning, etc. In terms of networks, BFs have applications in web caching [3]–[5], sensor networks [6]–[8], data center networks [9]–[11], cloud computing [12]–[14], and more [15]–[17]. In terms of databases, BFs have applications in key-value stores [18]–[23], privacy-preserving record linkage [24]–[29], block chain [30]–[32], and more [33]–[35]. In addition to the above, BFs also have applica-
tions that cannot be underestimated in data mining and analysis [36–42], and machine learning [43–47]. Most applications with set membership query can potentially be optimized using BFs [48].

Although BFs have been widely used in many applications, the fundamental issue of how to calculate FP probability remains elusive. Properly calculating the FP probability of BF is critical because it is used to calculate the optimal value of the important parameter \( k \), the number of hash functions. In [1], Bloom gave a formula for calculating the FP probability with known parameters \( n, k \), and \( m \). Based on Bloom’s formula, we can also easily compute the optimal value of the parameter \( k \) when \( m \) and \( n \) are known. This formula has been believed to be correct until 2008 when Prosenjit Bose et al. pointed out that Bloom’s formula is flawed and gave a new FP formula [49]. Interestingly, two years later, Ken Christensen et al. pointed out that Bose’s formula is also flawed and gave a new FP formula [50]. So far, it is believed that Christensen’s formula is perfectly accurate. However, Both Bose’s and Christensen’s FP formulas are also flawed and gave a new FP formula [51]. So far, it is believed that Christensen’s formula is perfectly accurate. However, Both Bose’s and Christensen’s FP formulas are too complicated to calculate the optimal value of \( k \) from given values of \( n \) and \( m \).

1.2 Main Contributions

While the conventional wisdom is to derive the optimal value of BF parameter \( k \) from the FP probability, in this paper, we propose the first approach to calculating the optimal \( k \) without any FP formula. We first observe that for a BF with \( m \) bits and \( n \) elements, if and only if its entropy is the largest, its false positive probability is the smallest, according to information entropy theory. Based on this observation, our approach is to derive a formula for calculating the optimal \( k \) by letting the entropy equal to 1. We also propose another method to calculate FP probability by deriving the left and right limit expressions of FP probability. We prove that when \( m \) goes to infinity, the left and right limits are the same, which is essentially the FP probability. Interestingly, our derived FP formula is the same as Bloom’s formula in [1]. This deepens our understanding of Bloom’s formula: it is perfectly accurate when \( m \) is infinitely large, and it is practically accurate when \( m \) is sufficiently large.

In summary, we make three key contributions in this paper. First, we propose an information theoretical approach to calculating the optimal value of BF parameter \( k \) without calculating FP probability. Second, we propose a new upper bound which is much more accurate than state-of-the-art. When \( m \) is infinitely large, our upper bound becomes the same as the lower bound. This result formally proves that Bloom’s formula is practically accurate when \( m \) is sufficiently large. Third, we conducted experiments to validate our findings. In particular, we show that the error of Bloom’s formula is negligibly small when \( m \) is large. Furthermore, we release our source code of Bloom Filters at GitHub [51]. The rest of this paper proceeds as follows. In Section 2, we show the derivation of the optimal number of hash functions using the information entropy theory. In Section 3, we present a new upper bound of the false positive probability of Bloom Filters. In Section 5 we derive the bounds of correct rate of Counting Bloom Filters (CBFs) through our proposed formulas about Bloom Filters. In Section 6 we conduct experiments to evaluate the error of Bloom Filters. We conclude the paper in Section 7.

2 Prior Art on BF False Positives

In this section, we review the prior art on calculating the false positive probability of Bloom Filters. Table 1 summarizes the notations used in this paper.

### Table 1: Symbols frequently used in this paper

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>Set of elements</td>
</tr>
<tr>
<td>( m )</td>
<td>BF size</td>
</tr>
<tr>
<td>( n )</td>
<td>Number of elements in ( S )</td>
</tr>
<tr>
<td>( k^* )</td>
<td>Optimal number of hash functions</td>
</tr>
<tr>
<td>( f )</td>
<td>False positive probability</td>
</tr>
<tr>
<td>( f_{\text{bloom}} )</td>
<td>False positive probability calculated by Bloom</td>
</tr>
<tr>
<td>( f_{\text{base}} )</td>
<td>False positive probability calculated by Bose</td>
</tr>
<tr>
<td>( f_{\text{christ}} )</td>
<td>False positive probability calculated by Christensen</td>
</tr>
<tr>
<td>( f_{\text{true}} )</td>
<td>True false positive probability of BF</td>
</tr>
<tr>
<td>( \text{FP} )</td>
<td>False positive</td>
</tr>
<tr>
<td>BF</td>
<td>Bloom Filter</td>
</tr>
</tbody>
</table>

2.1 Bloom’s False Positive Formula

In 1970, Bloom calculated the false positive probability of a Bloom Filter as follows [1]. Given a set \( S \) of elements, let \( n \) be the number of elements in \( S \), \( k \) be the number of hash functions, and \( m \) be the number of bits in the Bloom Filter \( A \) constructed from set \( S \). In querying an element \( x \), the false positive happens when the Bloom Filter reports that \( x \in S \) (i.e., \( A[h_i(x)] = 1 \) holds for each \( 1 \leq i \leq k \)), but actually \( x \not\in S \). Consider an arbitrary bit \( A[b] \) in \( A \). For any element in \( S \) and any hash function \( h_i (1 \leq i \leq k) \), the probability that this element is not hashed to bit \( A[b] \) by \( h_i \) is \( 1 - 1/m \). As \( S \) has \( n \) elements and each element is hashed \( k \) times, the probability of \( A[b] = 0 \) is \( p' \):

\[
p' = \left( 1 - \frac{1}{m} \right)^{kn} \tag{1}
\]

Thus, the probability of \( A[b] = 1 \) is \( 1 - (1 - 1/m)^{kn} \). For any element \( x \not\in S \), the probability that the false positive happens for \( x \), i.e., the probability of \( A[h_1(y)] \land A[h_2(y)] \land \cdots \land A[h_k(y)] = 1 \), is calculated as follows:

\[
f_{\text{bloom}} = \left( 1 - \left( 1 - \frac{1}{m} \right)^{kn} \right)^k \tag{2}
\]

which can be approximated by \( (1 - e^{-n/k})^k \).
2.2 Bose’s Derivation

In 2008, Bose et al. pointed out that the last step in Bloom’s derivation is flawed. The formula for Bloom Filters as follows:

Although for each bit $A[h_i(x)] = 1$, $A[h_k(x)] = 1$, for any element $x \notin S$, the $k$ events $A[h_1(x)] = 1, A[h_2(x)] = 1, \cdots , A[h_k(x)] = 1$ are not actually independent [49].

For each bit $A[h_i(x)] = 1$ and $A[h_k(x)] = 1$, for any element $x \notin S$, the $k$ events $A[h_i(x)] = 1$, $A[h_k(x)] = 1$, for any element $x \notin S$, the $k$ events $A[h_i(x)] = 1, A[h_k(x)] = 1, \cdots , A[h_k(x)] = 1$ are not actually independent [49].

2.4 Grandi’s Derivation

In 2018, Grandi proposed the $\gamma$-transform [52] approach to analyze the false positive probability [53], and further derived the false positive formula as follows:

$$f_{grandi} = \sum_{x=0}^{m} \left( \frac{x}{n} \right)^k \sum_{j=0}^{x} \left( \frac{x-j}{n} \right)^{kn} \cdot (x-j)^{kn}$$

(8)

where $x$ represents the number of bits set to 1 in the Bloom Filter.

3 Computing the Optimal K

Traditionally, the optimal number of hash functions is derived through finding the extrema of the asymptotic formula of $f_{bloom}$ given in Eq. 2 as follows:

$$k_{bloom}^* = \frac{m}{n} \ln 2$$

(9)

However, as we have discussed, the underlying formula for FP probability is not fully correct. In this section, we first present the important theorems that correlate information entropy and the false positive rate of Bloom Filters. Then we propose a method of deriving the optimal value of $k$ by minimizing the FP probability given the value of BF size $m$ and number of inserted elements $n$.

3.1 Information Entropy Basis

Information Entropy: In information theory, information entropy is used to measure the uncertainty of a random variable. In this paper, we use Shannon entropy [54], which measures the value of the information contained in a variable. Entropy is typically measured in bits, nats, or bans [55]. For a variable with $s$ events with the probabilities of $p_1, p_2, \cdots , p_s$. The information entropy $E$ is defined as:

$$E = -\sum_{i=1}^{s} p_i \log_2 \frac{1}{p_i}$$

(10)

Property of Information Entropy: For any variable or message, if its information entropy is not at the maximum, it can be compressed without information losses. For a random variable, the larger the uncertainty is, the bigger the information entropy is. Suppose an $m$-bit string variable is compressed to $m'$-bit string variable without loss of information, the entropy $E$ of the Bloom Filter remains the same. However, the average information containing in each bit, which is $e = E/m$, will increase to $e' = E/m'$. Then, we have $me = me'$. According to the information entropy formula (Eq. 10), we can obtain the information entropy $e'$ of one bit of a BF as follows, where $p'$ is defined in Eq. 1.

$$e' = -(p' \log_2 p' + (1-p') \log_2 (1-p'))$$

(11)

We illustrate the relationship between entropy $e'$ and $p'$ in Figure 1. We can observe that the entropy of an
3.2 An Important Global Assumption

What must be said before our formal analysis is that we assume after compression without information loss, a Bloom Filter is still a Bloom Filter. In other words, we assume the properties of query and insertion of the Bloom Filter should be preserved. Of course, that's a very strong assumption, and we have not given a rigorous proof. However, since the results of our analysis based on this assumption are in perfect agreement with our experiments, we state it here as an open question.

3.3 Relation Between False Positive Probability and Information Entropy

We now show the relation between the false positive probability and information entropy. We first present a lemma and some definitions.

**Lemma 1.** Given a BF, suppose $n$ and $k$ keep unchanged, when $m$ becomes larger, the FP probability gets smaller.

**Proof.** When $n$ and $k$ are fixed, for larger $m$, the probability of each bit being 1 in the BF becomes smaller, i.e., $p'$ becomes smaller; thus, the FP probability gets smaller.

**Definition 1.** Bloom Filter variable. The false positive rate of Bloom Filters is determined by $m$, $n$, and $k$. For different $n$ elements, the $m$-bit string varies. Thus, when the values of $m$, $n$, $k$ are given, the $m$-bit string is a random variable (similar but slightly different from the mathematical definition of random variable). When the $n$ elements are given, the $m$-bit string is a random variable instance. Therefore, when the values of $m$, $n$, $k$ are given, we call it a Bloom Filter Variable (BFR). Since it is a random variable, we can compute its information entropy.

**Definition 2.** Equivalent Bloom Filter variables. Given two Bloom Filters variables $v_1$ and $v_2$, for the same $n$ elements, there are a pair of BFR instances. Given a set with $n$ elements, if these pairs of BFR instances always report the same result: true or false for any input element, we say $v_1$ and $v_2$ are equivalent.

**Theorem 1.** Given a Bloom Filter variable $v_1$ with parameters $m$, $n$, and $k$, if its information entropy is not at the maximum, there must exist a smaller equivalent Bloom Filter variable $v_2$ with parameters $m'$, $n$, and $k$, where $m' < m$.

**Proof.** For $v_1$, the parameters are $m$, $n$, $k$. Suppose its $k$ hash functions are $h_1(\cdot), h_2(\cdot), \ldots, h_k(\cdot)$. Since the assumption is that the information entropy of $v_1$ is not at the maximum, according to the property of information entropy, $v_1$ can be compressed without information loss. After compression, suppose the new random variable has a length of $m'(m' < m)$, we name it $v_2$. Note that during the compression, the information value $m \ast e$ keeps unchanged; thus, the length of the compressed message has a minimum value. Here $v_1$ and $v_2$ are two variables consisting of bits, and we can treat them as two integer variables. We use $|In(v_1)|$ and $|In(v_2)|$ to represent the integer value of $v_1$ and $v_2$. Furthermore, we use $|In(v_1)|$ represents the length of $v_1$, then we have $|In(v_1)| = m$, $|In(v_2)| = m'$. Because we compress $v_1$ and get $v_2$, this can be regarded as a function $g(\cdot)$. In other words, $g(|In(v_1)|) = |In(v_2)|$. We can also obtain $v_1$ by equation $In(v_1) = g^{-1}(|In(v_2)|)$.

At this stage, we consider the new Bloom Filter variable $v_2$, the parameters are $m'$, $n$, and $k$. Note that we use different hash functions, and the $k$ hash functions are

\[
g^{-1}(|In(v_2)|) \ll h_1(y) \gg (|g^{-1}(|In(v_2)|)| - h_1(y) - 1),
\]

\[
g^{-1}(|In(v_2)|) \ll h_2(y) \gg (|g^{-1}(|In(v_2)|)| - h_2(y) - 1),
\]

\[
\ldots
\]

\[
g^{-1}(|In(v_2)|) \ll h_k(y) \gg (|g^{-1}(|In(v_2)|)| - h_k(y) - 1)
\]

(12)

where ‘$\ll$’ represents the left shift operator, and ‘$\gg$’ represents the right shift operator.

Given an input element $y$, we can compute the above $k$ values only using $v_2$ and $h_i(\cdot)$ without $v_1$. Then we need to prove that for any incoming element $y$, $v_2$ reports the same $k$-bit value. With formula (12) we use equation $v_1 = g^{-1}(|In(v_2)|)$ and $m = |g^{-1}(|In(v_2)|)|$. These $k$ hash functions are simplified as

1. The symbols ‘$\ll$’ and ‘$\gg$’ shift all the binary bits of a number to the left/right by several bits, respectively. On the premise that the number does not overflow, shifting left by $x$ bits is equivalent to multiplying by 2 to the power of $x$, and shifting right by $x$ bits is equivalent to dividing by 2 to the power of $x$. For example, shifting the integer 6 (‘110’ in binary) to the left by 1 bit is equivalent to the integer 12 (‘1100’ in binary), and shifting the integer 6 to the right by 1 bit is equivalent to the integer 3 (‘0011’ in binary).
Here \(1 \leq i \leq k\) and \(0 \leq h_i(y) \leq m - 1\). Here \(v_1 \ll h_1(y) \gg (m - h_1(y) - 1)\) actually means the value of \(h_i(y)\)-th bit of \(v_1\). This is the same as the \(k\) hash functions as \(v_1\). Therefore, \(v_1\) and \(v_2\) are equivalent.

**Theorem 2.** Given a Bloom Filter variable, if and only if its information entropy is at the maximum, its FP probability is at the minimum.

**Proof.** First, we prove that if the FP probability is at the minimum, then its average information entropy (Eq. 11) must be at the maximum. Given a Bloom Filter variable \(v_1\) with parameters \(m, n, k\). Since the assumption is that the average information entropy of \(v_1\) is not at the maximum, according to Theorem I, there exists a smaller Bloom Filter variable \(v_2\) with parameters \(m', n, k\). Since \(v_2\) and \(v_1\) are equivalent, their FP probabilities are the same, we name it \(f\). At this stage, we enlarge the size of \(v_2\) a little from \(m'\) to \(m''\), where \(m' < m'' < m\). According to Lemma 3, we know the FP probability of \(v_2\) becomes smaller than \(f\). This means that for \(v_1\), there exists a BF variable with a smaller size and a smaller FP probability. Therefore, the FP probability of \(v_1\) is not at the minimum. This means that if its average information entropy is not at the maximum, then the FP probability is definitely not at the minimum. The contrapositive is that if the FP probability is at the minimum, then its average information entropy must be at the maximum.

Second, we prove that if its average information entropy is at the maximum, the FP probability is at the minimum. Given a Bloom Filter variable \(v_1\) with parameters \(m, n, k\). Since the assumption is that the FP probability of \(v_1\) is not at the maximum, there exists an optimal Bloom Filter variable \(v_0\) with parameters \(m, n, k'\), where \(k' \neq k\), the average information entropy of \(v_0\) is at the maximum. According to Eq. 11 and Figure 1, the \(p'\) of \(v_0\) is 0.5 whereas the \(p'\) of \(v_1\) is not because they have different value of \(k\). Therefore, the average information entropy of BF1 is not at the maximum. This means if the FP probability is not at the minimum, its average information entropy must be not at the maximum. The contrapositive is that if its average information entropy is at the maximum, the FP probability is at the minimum.

3.4 Computing the Optimal \(k\)

According to Theorem 2, when the average information entropy of the Bloom Filter variable is at the maximum, the FP probability is at the minimum. Recalling the definition of \(p'\) in Eq. 11 one can use this interpretation to find \(k^*\), i.e., the optimal number of hash functions. From Figure 1 we know that when \(p' = 0.5\), \(E\) reaches the maximum value 1. By setting the value of \(p'\) to 0.5, we have

\[
p' = \left(1 - \frac{1}{m}\right)^k = 0.5
\]

Further, we have:

\[
k^* = -\frac{\ln 2}{n} / \ln \left(1 - \frac{1}{m}\right)
\]

This formula is very close to the formula of \(k^*\) obtained by Bloom. When \(x\) is very small, \(\ln(1 + x) \approx x\), and therefore \(-1/\ln(1 - 1/m) \approx m\), resulting the same term as in Eq. 9.

**Theorem 3.** Given any BF variable, when \(m\) and \(n\) are fixed, the FP probability \(f\) is a function of \(k\), we represent it \(f(k)\). Then \(f(k)\) is a well-defined function which has only one minimum value.

**Proof.** Given a Bloom Filter variable \(v_1\) with parameters \(m, n, k_1\), its entropy is \(E_1\). Given another Bloom Filter variable \(v_2\) with parameters \(m, n, k_2\), its entropy is \(E_2\).

1. For any \(k_1 < k_2 \leq k^*\), according to Eq. 1, Eq. 11 and Figure 1 we know \(p'_1 \leq p'_2 \leq 0.5\). We compress \(v_1\) to \(v_3\) with parameters \(m_3, n, k_1\). To make \(v_3\)'s entropy equal to \(E_2\), \(m_3\) should be \(mE_1/E_2\). In this case, the entropy of \(v_3\) is equal to that of \(v_2\). When the entropy of BF variables is less than 0.5, the same entropy leads to the same \(p'\). In other words, \(p'_3 = p'_2\). Because \(v_2\) has more hash functions \((k_2 > k_1)\), the FP probability of \(v_2\) is smaller than that of \(v_3\). While \(v_3\) and \(v_1\) have the same FP probability, therefore, the FP probability of \(v_2\) is smaller than that of \(v_1\). In other words, for any \(k\) \((k < k^*)\) increasing, the FP probability of BF decreases. (2) For any \(k^* \leq k_1 < k_2\), according to Eq. 1, Eq. 11 and Figure 1 we know \(p'_1 > p'_2 \geq 0.5\). Using the similar derivation, we can derive that the FP probability of \(v_2\) is larger than that of \(v_1\). According to the above two cases, we know that given any BF variable, when \(m\) and \(n\) are fixed, the FP probability of \(v\) is a well-defined function, \(f(k)\), which has a unique minimum value.

4 Asymptotic Form of the FP Probability

In this section, we derive a new approach to computing the asymptotic form for the FP probability of BF. The new derivation is based on partitioned Bloom Filters (pBF) that are used frequently to carry out parallel queries. Its underpinning principle is simple: the BF is divided into \(k\) even partitions, and each hash function only acts on one of the partitions, respectively. The probability that one bit of the BF array remains 0 after inserting \(n\) elements in the BF becomes the following as new each hash maps into \(\frac{n}{k}\) separate bits.

\[
p'_\text{partition} = \left(1 - \frac{k}{m}\right)^n
\]
It is intuitive that the FP probability of partitioned BF is a little bigger than that of BF. Unfortunately, there is no strict proof. Here we show one proof method, which is based on the following Lemma.

**Lemma 2.** For $m > 1$, $k > 1$, $n > 1$, $m > k$,

$$
\left(1 - \frac{k}{m}\right)^n < \left(1 - \frac{1}{m}\right)^{kn}
$$

(17)

**Proof.** Directly application of Bernoulli inequality. \(\square\)

The above lemma shows that $p_{partition}' < p_{true}'$ or equivalently $1 - p_{partition}' > 1 - p_{true}'$. Thus, we know that with the same parameters, the FP probability of the pBF will be larger than that of the standard BF $f_{true}$, i.e., $f_{partition} > f_{true}$. In addition, Bose’s bounds in Eq. 5 state that the precise value of FP probability for a BF $f_{true}$ is larger than $f_{bloom}$, i.e., $f_{true} > f_{bloom}$. Therefore, we have the following upper and lower bounds:

$$
f_{partition} > f_{true} > f_{bloom}
$$

(18)

For a partitioned BF, the probability that one bit of the array is still 0 $p'$ is shown in Eq. 16. Different from standard Bloom Filters, for a partitioned Bloom Filter, the event $E(h_1 = 1), E(h_2 = 1), E(h_3 = 1), \ldots, E(h_{i-1} = 1)$ is independent of the event $E(h_{i-1} = 1)$, where $E(h_{i-1} = 1)$ means that the event that the position of $h_{i-1}(x)$ is 1 because each hash function is responsible for one partition, and has no impact on each other. Therefore, we have

$$
f_{partition} = (1 - p_{partition}')^k = \left(1 - \left(1 - \frac{k}{m}\right)^n\right)^k
$$

(19)

Then, the formula (18) becomes

$$
\left(1 - \left(1 - \frac{k}{m}\right)^n\right)^k > f_{true} > \left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^k
$$

(20)

Then, we use the well-known limit formula:

$$
\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^{-x} = e
$$

(21)

Asymptotically, when $m$ becomes large, we already know that $f_{bloom}$ converges to the term in Eq. 2 for $k = 1$. Nevertheless, the upper bound has also an asymptotic behaviour as the following, which is the same term as the lower bound limit.

$$
\lim_{m \to \infty} \left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^k = \left(1 - e^{-nk/m}\right)^k
$$

(22)

Through the Sandwich Theorem (also known as squeeze theorem) we obtain the following equation, which is similarly to Christensen and Bose:

$$
\lim_{m \to \infty} f_{true} = \left(1 - e^{-nk/m}\right)^k
$$

(23)

This means that when $m$ is large, the Bloom’s formula can be used with negligible error. However, we still need to evaluate what means $m$ being large. We will do this by comparing the two bounds we have in hand: the one from Bose and the one we derived in this paper. We show in Figure 2 that the two upper bounds along with the lower bound obtained for $k = 7$ and $m = 10n$ as a function of $n$, the number of elements inserted in the BF. As can be seen, the upper bound derived in this paper and the lower bound $f_{bloom}$ converge relatively fast for $n = 9$, while the upper bound derived by Bose has a much slower convergence. We can see this better by looking at the behavior of the bounds error ratio $\beta$, defined as $\beta = \frac{upper\ bound - lower\ bound}{lower\ bound}$, for the two bounds in Figure 3.

As can be seen, the gap between our derived upper
bound and $f_{\text{bloom}}$ is decreasing polynomially at a constant speed, while Bose’s bound has a lower speed of convergence. In order to extend this observation, we show in Figure 4 the evolution of the bounds error ratio for a BF with $m = 10000$, $n = 1000$ and varying $k$.

As expected, error involved with using $f_{\text{bloom}}$ increases with the number of hash functions $k$ increases. However, it can be seen that the convergence behavior of the bounds derived in this paper is much better than the one obtained by Bose.

5 Correct Rate of Counting Bloom Filters

The Counting Bloom Filter (CBF) [3], one widely used variant of standard Bloom Filter, replaces each bit with one counter, supporting estimating the frequency of each element in a multiset. Specifically, given a multiset $S$ of $n$ distinct elements with their corresponding frequencies, we create a counter array $A$ of length $m$ as follows. First, we initialize each counter of $A$ to 0, then for each element $x \in S$, we use $k$ hash functions to compute $k$ hash values: $h_1(x), h_2(x), \ldots, h_k(x)$ where each hash value is in the range $[1, m]$. Second, for each $1 \leq i \leq k$, we let $A[h_i(x)] = A[h_i(x)] + 1$. Let $f_x$ be the frequency of element $x$ in multiset $S$. Therefore, the step of $A[h_i(x)] = A[h_i(x)] + 1$ for each $1 \leq i \leq k$ will occur $f_x$ times. The resulting counter array $A$ is called the CBF for multiset $S$. To query the frequency of an element $y$ in multiset $S$, we first use the same $k$ hash functions to compute $k$ hash values: $h_1(y), h_2(y), \ldots, h_k(y)$. Second, we report the minimum value of the $k$ counters: $A[h_1(y)], A[h_2(y)], \ldots, A[h_k(y)]$ as the estimated frequency of this element. Obviously, the estimated frequency reported by the CBF is always larger than or equal to the real frequency for any element in multiset $S$. The case that the estimated frequency from the CBF is equal to the real frequency for one element is called the correct case. The probability of such case happening is called the correct rate of the CBF ($C_r$).

The calculation of the correct rate of CBFs can benefit from our derivation of the false positive probability of standard Bloom Filter. In querying an element $x$, the correct case happens when there exists at least one hashed counter (among $A[h_1(x)], A[h_2(x)], \ldots, A[h_k(x)]$) that is not hashed by any elements in multiset $S \setminus \{x \cdot f_x\}$. The contrapositive is that the correct case does not happen when all the $k$ hashed counters are also hashed by some elements in multiset $S \setminus \{x \cdot f_x\}$. Consider an arbitrary counter $A[b]$ in $A$. For any distinct element in $S \setminus \{x \cdot f_x\}$ and any hash function $h_i$ ($1 \leq i \leq k$), the probability that this element is not hashed to counter $A[b]$ by $h_i$ is $1 - 1/m$. As $S \setminus \{x \cdot f_x\}$ has $n - 1$ distinct elements and each distinct element is hashed $k$ times, the probability that $A[b]$ is not hashed by any element in multiset $S \setminus \{x \cdot f_x\}$ is $p_c$:

$$p_c = \left(1 - \frac{1}{m}\right)^{k(n-1)}$$

(24)

Thus, the probability that $A[b]$ is hashed by some elements in multiset $S \setminus \{x \cdot f_x\}$ is $1 - (1 - 1/m)^{k(n-1)}$. The probability that all the $k$ hashed counters are also hashed by some elements in multiset $S \setminus \{x \cdot f_x\}$ is answered by $f_{\text{true}}$ with element number of $n - 1$. We denote $f_{\text{true}}|_{n-1}$ as $f_{\text{true}}$ with element number of $n - 1$, and get:

$$1 - C_r = f_{\text{true}}|_{n-1} \Rightarrow C_r = 1 - f_{\text{true}}|_{n-1}$$

(25)

Applying Eq. [20], we can get the upper and lower bounds of the $C_r$ of CBFs:

$$1 - \left(1 - \left(1 - \frac{k}{m}\right)^{n-1}\right)^k < C_r < 1 - \left(1 - \left(1 - \frac{1}{m}\right)^{(n-1)k}\right)^k$$

(26)

6 EXPERIMENTAL RESULTS

In this section, we first validate our proposed formula of optimal $k$ (Eq. [15]). Second, we compare our proposed upper bound of the FP probability of BFs (Eq. [20]) with Bose’s upper bound (Eq. [5]). Third, we validate our proposed upper and lower bounds of the correct rate of CBFs (Eq. [26]).

6.1 Experimental Setup

Datasets: We use four kinds of datasets: 1) IP Trace Dataset; 2) Data Center Dataset; 3) Network Dataset; 4) Synthetic Dataset. The details are as follows.

1) IP Trace Dataset: The IP Trace Dataset contains anonymized IP traces collected in 2016 by CAIDA [56]. Each item contains a source IP address (sip, 4 bytes) and a destination IP address (dip, 4 bytes), 8 bytes in total. We treat each sip-dip pair as an ID.

2) Data Center Dataset: The Data center dataset [57] contains traces collected from the data centers in [58]. Each item (4 bytes) represents the ID of the trace.
3) **Network Dataset:** The network dataset contains users’ posting history on the stack exchange website [59]. Each item (4 bytes) represents the ID of each user.

4) **Synthetic Dataset:** We generate a synthetic dataset that follows the Zipf [60] distribution using Web Polygraph [61], an open source performance testing tool. The length of each item ID is 4 bytes, and the skewness of the dataset is 1.5.

**The operations on the above datasets:** For BF, we regard each distinct ID as an element in the aforementioned set. We use (part of) the first $100/1.5/0.17/0.23M$ distinct IDs to construct BFs, and query the next $300/4.5/0.51/0.69M$ distinct IDs to get the empirical FP probabilities of these BFs. For CBF, we regard each distinct ID and its occurrence number as a distinct element and the corresponding frequency in the aforementioned multiset, respectively. We use (part of) the first $100/1.5/0.17/0.23M$ distinct IDs and their occurrence numbers in the current trace to construct CBFs, and query their frequencies to get the empirical correct rates of these CBFs.

**Implementation:** We have implemented the standard Bloom filter in C++. We use the Bob Hash (obtained from the open source website [62]) with different initial seeds to implement the hash functions in BFs as recommended by literature [63]. All the implementation source code is made publicly available at GitHub [4].

### 6.2 Optimal $k$ Formula Validation

#### 6.2.1 Optimal $k$ vs. $n$

Figure 5(a)-5(d) plot the empirically and theoretically optimal $k$ with different $n$ for $m = 500M$, 7.5M, 0.85M, 1.15M. Our results show that the optimal $k$ calculated from our new formula follows the empirically optimal $k$ very well, regardless of the values of $n$. We observe that the optimal $k$ calculated from our new formula is very close to the one calculated from the formula obtained by Bloom.

#### 6.2.2 Optimal $k$ vs. $m$

Figure 6(a)-6(d) plot the empirically and theoretically optimal $k$ with different $m$ for $n = 1.5M$, 7.5M, 0.85M, 0.115M. Our results show that the optimal $k$ calculated from our new formula follows the empirically optimal $k$ very well, regardless of the values of $m$. We observe that the optimal $k$ calculated from our new formula is very close to the one calculated from the formula obtained by Bloom, especially when $m$ becomes larger.

### 6.3 Upper Bound Comparison

#### 6.3.1 Upper Bound vs. $n$

Figure 7(a)-7(d) plot the empirical results, Bloom’s theoretical results, Bose’s upper bounds, and our upper bounds of FP probability with different $n$ for $m = 500M$, 7.5M, 0.85M, 1.15M and $k = 6$. Our results show that our upper bounds of FP probability follows the empirical FP probability very well, regardless of the values of $n$. We find that all above four results almost coincide with each other, which demonstrates the tightness of bounds in Eq. 5 and 20. To compare upper bound of Bose and ours more intuitively, in Figure 8(a)-8(d) we plot the bounds error ratio $\beta$, defined as $\beta = \frac{\text{upper bound} - \text{lower bound}}{\text{lower bound}}$, of these two upper bounds with different $n$ for $m = 500M$, 7.5M, 0.85M, 1.15M and $k = 6$. Our results show that the bounds error ratio of our upper bound is 23070.7 times lower than that of Bose’s upper bound on average. We find that our upper bound almost coincides with the lower bound, which demonstrates the superiority of our upper bound.

#### 6.3.2 Upper Bound vs. $m$

Figure 9(a)-9(d) plot the empirical results, Bloom’s theoretical results, Bose’s upper bounds, and our upper bounds of FP probability with different $m$ for $n = 50M$, 0.75M, 0.085M, 0.115M and $k = 6$. Our results show that our upper bounds of FP probability follows the empirical FP probability very well, regardless of the values of $m$. Figure 10(a)-10(d) plot the bounds error ratios of these two upper bounds with different $m$ for $n = 50M$, 0.75M, 0.085M, 0.115M and $k = 6$. Our results show that the bounds error ratio our upper bound is 1883.9 times lower than that of Bose’s upper bound on average. We find that our upper bound almost coincides with the lower bound, regardless of the values of $m$.

#### 6.3.3 Upper Bound vs. $k$

Figure 11(a)-11(d) plot the empirical results, Bloom’s theoretical results, Bose’s upper bounds, and our upper bounds of FP probability with different $k$ for $n = 50M$, 0.75M, 0.085M, 0.115M and $m = 500M$, 7.5M, 0.85M, 1.15M. Our results show that our upper bounds of FP probability follows the empirical FP probability very well, regardless of the values of $k$. Figure 12(a)-12(d) plot the bounds error ratios of these two upper bounds with different $k$ for $n = 50M$, 0.75M, 0.085M, 0.115M and $m = 500M$, 7.5M, 0.85M, 1.15M. Our results show that the bounds error ratio of our upper bound is 14571.0 times lower than that of Bose’s upper bound on average.

### 6.4 $C_r$ Formula Validation

Figure 13(a)-13(d) Figure 14(a)-14(d) and Figure 15(a)-15(d) plot the correct rates of CBFs with different values of $n$, $m$, and $k$, respectively. Our results show that our lower and upper bounds of the correct rate of CBFs follow the empirical correct rates very well, regardless of the values of $n$, $m$, and $k$.

### 7 Conclusion

In this paper, we discuss the evolutionary of the formula of Bloom Filter. Three formulas of false positive probability are presented: Bloom’s formula, Bose’s formula, and Christensen’s formula. There is an error in the deduction process of Bloom’s formula, and a minor error in Bose’s formula. Christensen’s formula is correct, but the false positive must be calculated using an iterative table-based
algorithm with a time complexity of $O(knm)$. What is worse, it cannot deduce the optimal value of $k$. To compute the optimal value of $k$, we use information and entropy theory to deduce the exact formula of $k$. To compute the false positive probability, 1) For small $m$, Christensen's formula can be used; 2) For large $m$, we propose a new upper bound which is much more accurate than state-of-the-art. Fortunately, when $m$ is infinitely large, our upper bound becomes the same as the lower bound, which is Bloom's formula. Besides, we
derive the bounds of correct rate of Counting Bloom Filters through our proposed formulas about Bloom Filters. All the implementation source code is made publicly available at GitHub [51].

Fig. 9: FP probability vs. $m$ for $n = 50M, 0.75M, 0.085M, 0.115M$ and $k = 6$.

Fig. 10: Bounds error ratio vs. $m$ for $n = 50M, 0.75M, 0.085M, 0.115M$ and $k = 6$.

Fig. 11: FP probability vs. $k$ for $n = 50M, 0.75M, 0.085M, 0.115M$ and $m = 500M, 7.5M, 0.85M, 1.15M$.

Fig. 12: Bounds error ratio vs. $k$ for $n = 50M, 0.75M, 0.085M, 0.115M$ and $m = 500M, 7.5M, 0.85M, 1.15M$.

**ACKNOWLEDGMENT**

We would like to thank Siyuan Dong for his participation in the experiments, and Yuhan Wu and Ruwen Zhang for their participation in the discussions. We would like to...
thank the anonymous reviewers for their valuable suggestions and comments. This work is supported by Key-Area Research and Development Program of Guangdong Province 2020B0101390001, National Natural Science Foundation of China (NSFC) (No. U20A20179, 61832001).

REFERENCES


Zhuochen Fan is currently pursuing the Ph.D. degree with School of Computer Science, Peking University, advised by Tong Yang. His research interests include network big data, network measurements, streaming algorithms, Bloom Filters, and mobile computing.

Gang Wen received the B.S. degree in mathematics from Peking University in 2022. His research interests include network big data, network measurements, streaming algorithms, Bloom Filters, and deep learning theory. He has received the Excellent Academic Student Award and Scholarship for two years from Peking University. Currently, he is a Ph.D. student majoring in statistics at Yale University.

Zhipei Huang received the M.S. degree from Peking University, advised by Tong Yang. He is interested in applying machine learning to indexing, computer networks and data stream processing.

Yang Zhou graduated from Peking University, advised by Tong Yang. He is currently pursuing the Ph.D. degree with Harvard University, advised by Minlan Yu. He is broadly interested in streaming algorithms, networked systems, and data-intensive systems.

Qiaobin Fu received the B.E. degree from the Dalian University of Technology in 2011 and the M.S. degree from the University of Chinese Academy of Sciences in 2014. He earned his Ph.D. degree from Boston University in 2020. Currently, he is working at Google on network-aware scheduling and cross-resource optimization. His research interests focus on computer networking, inter/intra-data center networks, and cloud computing.

Tong Yang (Member, IEEE) received the Ph.D. degree in computer science from Tsinghua University in 2013. He visited the Institute of Computing Technology, Chinese Academy of Sciences (CAS). Currently, he is an Associate Professor with School of Computer Science, Peking University. His research interests include network big data, sketches, network measurements, Bloom Filters, IP lookups, KV stores, and hash tables. He published papers in SIGCOMM, NSDI, SIGKDD, SIGMOD, VLDB, ICDE, INFOCOM, USENIX ATC, TKDE, TC, TPS, ToN, VLDB Journal, JSAC, etc. He is currently an Associate Editor for Knowledge and Information Systems.

Alex X. Liu (Fellow, IEEE) received his Ph.D. degree in Computer Science from The University of Texas at Austin in 2006, and is currently the President of the Software Engineering Institute and the Chief Security Officer of the Midea Group. Before that, he was the Chief Scientist of the Ant Group, and further before that, he was a Professor of the Department of Computer Science and Engineering at Michigan State University. He received the IEEE & IFIP William C. Carter Award in 2004, a National Science Foundation CAREER award in 2009, the Michigan State University Withrow Distinguished Scholar (Junior) Award in 2011, and the Michigan State University Withrow Distinguished Scholar (Senior) Award in 2019. He has served as an Editor for IEEE/ACM Transactions on Networking and an Area Editor for Computer Communications. He is currently an Associate Editor for IEEE Transactions on Dependable and Secure Computing and IEEE Transactions on Mobile Computing. He has served as the TPC Co-Chair for ICNP 2014 and IFIP Networking 2019. He received Best Paper Awards from SECON-2018, ICNP-2012, SRDS-2012, and LISA-2010. His research interests focus on cybersecurity, cloud computing, dependable computing, and privacy-preserving computing. He is a Member of Academia Europaea, a Member of the European Academy of Sciences and Art, an IEEE Fellow, an IET Fellow, an AAIA Fellow, and an ACM Distinguished Scientist.

Bin Cui (Senior Member, IEEE) is a professor and Vice Dean in School of CS at Peking University. He obtained his Ph.D. from National University of Singapore in 2004. His research interests include database system architectures, query and index techniques, big data management and mining. He is serving as Vice Chair of Technical Committee on Database China Computer Federation (CCF) and Trustee Board Member of VLDB Endowment, is also in the Editorial Board of Distributed and Parallel Databases, Journal of Computer Science and Technology, and SCIENCE CHINA Information Sciences, and was an associate editor of IEEE Transactions on Knowledge and Data Engineering (TKDE) and VLDB Journal. He was awarded Microsoft Young Professorship award (MSRA 2008), CCF Young Scientist award (2008), Second Prize of Natural Science Award of MOE China (2014), and appointed as Cheung Kong distinguished Professor by MOE in 2016.